**Lesson 0**

Infimum and Supremum

<https://www.cantorsparadise.com/why-we-use-the-infimum-and-supremum-in-mathematics-32a90ba13c6c>

The infimum and supremum are concepts in mathematical analysis that generalize the notions of minimum and maximum of finite sets

**Closed Intervals**

Consider the closed interval of

The minimum is very clearly and the maximum is very clearly

The points of the interval are represented by x where



**Open Intervals**

Consider the closed interval of

The open interval consists of all the numbers between and , but not and themselves

No minimum for the interval exists

No maximum for the interval exists



While minima and maxima may not exist for the open interval , we’d like to identify the next best alternatives.

We call a the ‘infimum’ of the open interval , and b the ‘supremum’.

The supremum of a set is its least upper bound. It is the least element in that is greater than or equal to each individual element of .

suprema (sup).

The infimum of a set is its greatest upper bound. It is the greatest element in that is less than or equal to each individual element of .

infima (inf)

can be expressed as x where , we see that a is the infimum

**Lesson 0**

Limit inferior and limit superior

**limit superior**

Also

**limit inferior**

Also

Example: ASS1 Q4 a i

find the limit (continuous) superior/inferior of , as

**Wolframalpha**

lim n -> inf (sqrt(2n+1)-sqrt(2n))

conjugate:

reciprocal:

power rule:

Example: ASS1 Q4 a ii

find the limit (continuous) superior/inferior of , as

**Wolframalpha**

lim n -> inf ((3n^3-n+8)/(4n(n-1)(n-2))

divide by leading term

product rule:

reciprocal:

power rule:

limiting value of sequence:

**Lesson 0**

Converse, Inverse and Contrapositive

<https://www.varsitytutors.com/hotmath/hotmath_help/topics/converse-inverse-contrapositive>

|  |  |
| --- | --- |
| Statement | If , then |
| Converse | If , then |
| Inverse | If not , then not |
| Contrapositive | If not , then not |

**Example: where the hypothesis and conclusion are equivalent**

|  |  |
| --- | --- |
| Statement | If two angles are congruent,  then they have the same measure TRUE |
| Converse | If two angles have the same measure,  then they are congruent. TRUE |
| Inverse | If two angles are not congruent,  then they do not have the same measure. TRUE |
| Contrapositive | If two angles do not have the same measure,  then they are not congruent. TRUE |

**Example:**

|  |  |
| --- | --- |
| Statement | If a quadrilateral is a rectangle, then it has two pairs of parallel sides. TRUE |
| Converse | If a quadrilateral has two pairs of parallel sides, then it is a rectangle. FALSE |
| Inverse | If a quadrilateral is not a rectangle, then it does not have two pairs of parallel sides. FALSE |
| Contrapositive | If a quadrilateral does not have two pairs of parallel sides, then it is not a rectangle  TRUE |

**Lesson 0**

Proof by contradiction (Contrapositive Proof)

*AKA indirect proof*

**Remember our contrapositive statement:**

|  |  |
| --- | --- |
| Statement | If , then |
| Converse | If , then |
| Inverse | If not , then not |
| Contrapositive | If not , then not |

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Description automatically generated**

**Example: NOV 2014 Q1.2**

Use proof by contradiction to prove:

If is a rational number and is an irrational number,

then is an irrational number

[1] Suppose it’s not true that is an irrational number

So is a rational number

[2] Let be a rational number and be an irrational number

There exists an integer (with ) such that

and

Hence

Which is a rational number, which is a contradiction

Thus is a rational number.

Example

<https://www.people.vcu.edu/~rhammack/BookOfProof2/Contrapositive.pdf>

Use proof by contradiction to prove:

If

Then

[1] Suppose it’s not true that is not true

So

[2] *Multiply both sides of by the positive value .*

How did we choose ?

Hence

Therefore it is not true that

Example: From Real analysis

Use proof by contradiction to prove:

Let be a set of test scores.

If the average of this set of scores is greater than 90,

then

at least one of the scores is greater than 90.

Let

P = If the average of this set of scores is greater than 90,

then

Q = at least one of the scores is greater than 90.

[1] Suppose the statement is not true:

We want to show that

~Q: none of the scores is greater than 90.

then

~P: the average of this set of scores is not greater than 90,

[2] Suppose

The sum of which

The average will be

Thus the average is not greater than 90 which implies

~Q: at least one of the scores is greater than 90.

**Lesson 0**

Proof by Induction

(From COS2601 and MAT2612)

This essentially is the same as using a recursive definition to define a language. It uses a trick which allows you to prove a statement about an arbitrary number n by first proving it is true when n is 1 and then assuming it is true for n=k and showing it is true for n=k+1.

**Proof by induction:**

|  |  |
| --- | --- |
| Recursive Definition | Clause |
| Show n=1 is true (LHS = RHS) | Basis Clause |
| substitute k into formula (LHS) | Inductive Clause |
| Substitute k+1 into formula (RHS) | Extremal clause |

Example MAT2612 ASS 01 Q1a

Let be the statement

**Basis Clause**

Show that

is where

.

Therefore, is true

**Inductive Hypothesis**

Show that.

is where

Assume

**Extremal Clause**

If is true, then must also be true

Assume

But,

Therefore, by the induction hypothesis:

Thus, is true

Hence, is true

It then follows by mathematical induction that is true.

**Example: ASS1**

**Clause**

Let be the statement

for all

**Basis Clause**

Show that

is where

LHS=3

RHS=2+1=3

Therefore, is true

**Inductive Clause**

Show that

is where

Assume

for all

**Extremal Clause (Inductive Step)**

If is true then must also be true

Assume

*Substitute inductive clause RHS here*

*Make this statement as close to LHS here*

Thus

for all

**Lesson 0**

Functions

**Injective (no two or more A’s pointing to the same B)**

is one-to-one (short hand is ) or injective if preimages are unique. In this case,

Example:

A picture containing text, receipt, screenshot

Description automatically generated

*Show that*

Thus is injective

**Surjective (Onto, every B has at least a matching A)**

is onto or surjective if every has a preimage. In this case, the range of is equal to the codomain.

*Make x the subject of the equation*

*Add in any restrictions*

*Show that the new function is not equal to any other piecewise parts (i.e. )*

Thus for any we can find an with .

We also have when

Thus all the elements of on the y-axis has an element such that

Thus is surjective

**Bijective**

is bijective if it is surjective and injective (one-to-one and onto).

The function is surjective and injective.

*Find the inverse function*

The inverse function is defined by

**Lesson 0**

Sequences

<https://tutorial.math.lamar.edu/Classes/CalcII/Sequences.aspx>

**General Sequence**

– first term

– second term

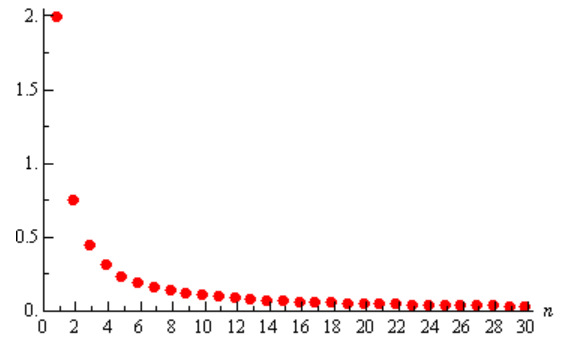
– term

– term

Denoted by:

Example: write down the first few terms of the following sequences

The graph:

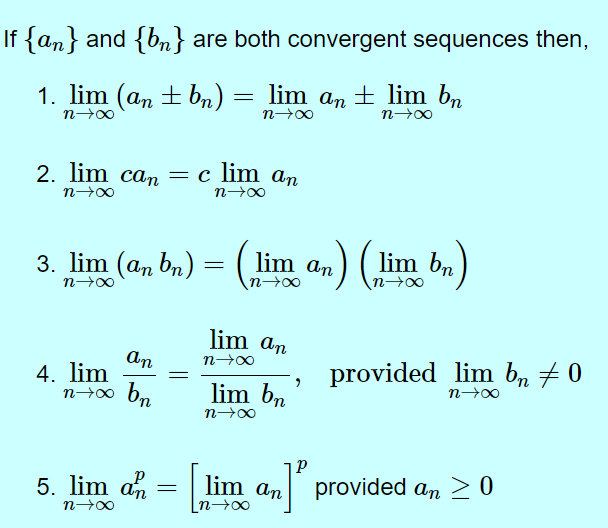


We then say that zero is the limit (or sometimes the limiting value) of the sequence.

limiting value of sequence:

**Convergent**

if exists and is finite



**Divergent**

if doesn’t exist or is infinite

example: determine if the following sequences converge or diverge

divide by leading term

the sequence converges and its limit is

**Lesson 0**

Monotone Sequences

<https://tutorial.math.lamar.edu/Classes/CalcII/MoreSequences.aspx>

Given the sequence

**Monotonic sequence**

Increasing sequence:

Decreasing sequence:

Lower bound (bounded below)

, for every

*There exists a number such that for every*

Upper bound (bounded above)

, for every

*There exists a number such that for every*

**Bounded Sequence**

If the sequence is both bounded below and bounded above

Example: Determine if the following sequences are monotonic and/or bounded

– first term for the first term

– second term for the second term

…

for every

the sequence is decreasing and monotonic.

Let

*Logically this is the only choice we can think of that fits*

Then for every

the sequence is bounded above

There are no numbers that exist that are lower than numbers in the sequence

the sequence is bounded above, but it is not bounded

Strictly Increasing

for all

Strictly Increasing

for all

**Lesson 0**

definition of the limit of a sequence

Definition: A real number is said to be a limit of a sequence if,

and only if

such that

for all

Example: ASS1 Q5 a.

Show by using an argument that

[1] re-write statement in terms of

[2] calculate and

Let be given.

Let be given.

[3] calculate

Let

Wolframalpha

abs(3Power[n,2]-2n+1/2Power[n,2]-4-3/2)

Numerator:

Denominator:

*add back constant*

for all

Thus

*add back constant?*

for all

[4] Let

Given

If , then

There must exist some

[5]

*For every positive real number*

*There exists a natural number such that*

Therefore by the Archimedean principle,

there exists with

*There exists a natural number such that*

Let

Thus we have

since

since

Which proves convergence

**Lesson 0**

definition of continuity

<https://www.kent.ac.uk/smsas/personal/elm2/MA552/MA552Worksheet5.pdf>

Definition: Let and .

is continuous at if for every

there exists such that implies .

Example: Prove that is continuous at .

Let be given.

Let .

Then for we have

Since , and we have ,

so .

Therefore,

**Lesson 0**

Convergent Series

For a series to converge (or exist), then

If the value is any other real number other than or the limit does not exist, the series diverges.

Example: ASS2 2020 Q2a

In each of the following, and whether the given series is convergent or divergent and where appropriate conditionally or absolutely convergent.

Step 1: rewrite the equation in terms of

*Remember to write all in terms of*

Let

Then

Step 2: find the limit

**wolframalpha**

Limit[abs((-Power[3n,2]+2)/Power[4n,2])),n->∞]

*Remember brackets!!*

*Product rule*

*Evaluate limits as they tend to*

Step 3: if , the series diverges

if , the series converges

Since

Which implies

Then from the contrapositive of the vanishing condition the given series is divergent and no further testing is required.

Example: ASS2 2021 Q2a

In each of the following, and whether the given series is convergent or divergent and where appropriate conditionally or absolutely convergent.

Step 1: rewrite the equation in terms of

*Remember to write all in terms of*

Let

Then

Step 2: find the limit

**wolframalpha**

Limit[abs(1/sqrt[n(n+1)]),n->∞]

*Remember brackets!!*

Step 3: p-series test

if , the series diverges

if , the series converges

Therefore the series is divergent

Step 4: limit comparison test

For series and where

If

Then either both series converge of both diverge

*try get a 1 in the numerator*

*remove constants in denominator*

Step 5: find

**wolframalpha**

Limit[(n/sqrt[n(n+1)]), n->∞]

=1

Therefore the series is also divergent

Therefore is also divergent (from the comparison test)

Step 6: test if series is decreasing i.e.

If , the series is conditionally convergent

If , the series is absolutely convergent

So

Thus the series is conditionally convergent

Lesson 0

Taylor Polynomials in

From MAT2615 Calculus in higher dimensions

This helps us obtain accurate approximations can be obtained by using higher order derivatives.

Example: Find the 4th degree Taylor polynomial for centered at and use it to approximate

*General formula for Taylor polynomial*

*Substitute into general formula*

*Therefore, approximation of*

Example: Find the 4th degree Taylor polynomial for and use it to approximate

*General formula for taylor polynomial where*

*Substitute into general formula*

*Therefore, approximation of*